A.A.S. **4**, I, 67-104 (2009)

MORTALITY IN IRELAND AT ADVANCED AGES, 1950-2006: PART 2: GRADUATED RATES

By S. F. Whelan

ABSTRACT

We graduate the Irish mortality experience from 1950 to 2003 by mathematical formulae from ages 75 years and upwards. The shape of the mortality curve at advanced ages is shown to be different to that recorded in the official tables, with the curve best fitted with Kannisto's version of Perks's Law. Mortality rates show only a modest trend of improvement in the early decades, below improvements in other developed countries. We evaluate the various approaches suggested to date to extend the method of extinct generations so mortality rates for non-extinct generations can be estimated. It is shown that the key advantage of this method is not in correcting for age misstatements but in achieving a close correspondence between death counts and the exposed to risk. This insight allows a rather straightforward approach to estimating the mortality of non-extinct generations. Applying the approach, we show that there has been an acceleration in the rate of improvement in more recent decades, but secular improvements in Irish mortality at advanced ages still lag behind those of England and Wales.

KEYWORDS

Irish Mortality; Method of Extinct Generations; Graduation; Mortality Laws; Perks's Law; Beard's Law; Kannisto's Law; Makeham's Law; Heligman–Pollard Formulae

CONTACT ADDRESS

Shane Whelan, School of Mathematical Sciences, University College Dublin, Ireland. Tel: +3531-7167155; Fax: +3531-7161186; E-mail: Shane.Whelan@ucd.ie

1. Introduction

The companion paper, *Mortality in Ireland at Advanced Ages, 1950-2006: Part 1: Crude Rates*, Whelan (2009), reconsidered the official Irish mortality record since 1950 as documented in the Central Statistics Office (CSO) Irish Life Tables. Several problems were identified that frustrate the accurate assessment of mortality rates at older ages, including age heaping in census counts and in reported age at death, and age exaggeration in census returns, especially at age 100 years. More materially, the errors in estimating the crude mortality rates due to a lack of correspondence between registered deaths and the exposed-to-risk from census returns were shown to grow in significance with advancing age. Finally, secular trends in mortality could not be reliably identified from the official Irish Life Tables because of the

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

inconsistency created by employing different models to graduate and extrapolate mortality rates at advanced ages from one life table to the next.

Mortality rates were re-estimated in *Part 1* using the method of extinct generations to overcome the significant errors introduced at older ages by the census method. The crude mortality rates using the method of extinct generations show that mortality in Ireland over the period from the 1950s to the 1980s was marginally lower than recorded in the Irish Life Tables, so life expectancies at advanced ages were higher than previously believed. In particular, the crude mortality rates did not increase as steeply with age as reported in the official tables. The re-estimated crude rates for both males and females show a very slight decrease in mortality rates up to age 90 years from the 1950s to the 1980s, with no improvement discernible at older ages. The improvements at advanced ages in Ireland have not been as great as those in England and Wales or other developed countries over the same period.

This paper graduates the crude rates using several popular formulae, different calibration techniques, and different evaluation criteria and extrapolates mortality curves to very advanced ages. The graduation approach recognises the remaining biases in the crude mortality rates arising from age heaping in death counts, particularly at ages 80 and 90 years. Extensions to the method of extinct generations so that mortality rates of more recent, still surviving generations, can be estimated are explored and the results reported.

The layout of the paper is as follows. The crude rates are graduated by mathematical formula in Section 2. After exploration with different formulae, parameter estimation approaches and evaluation criteria, the Kannisto model (a two-parameter version of Perks's Law) is found to be robust and outperform the other models when extrapolated to ages both below and above the fitted age range. Section 3 evaluates the various approaches suggested to date to extend the method of extinct generations so mortality rates for non-extinct generations can be estimated and proposes a new approach. It is shown that the key advantage of the method of extinct generations is not, as hitherto supposed, in correcting for age misstatements, but in achieving a closer correspondence between death counts and the exposed to risk. Accordingly, special care must be taken in adjusting the census count to constrain extensions to the method of extinct generations. Our study with Irish data suggests a straightforward and more transparent approach to extending the method of extinct generations to estimate the likely range of mortality rates at advanced ages in more recent times. We apply this new approach to provide estimates of Irish mortality at advanced ages up to 2001-2003. Section 4 concludes by summarising the results of our investigations, which shows a modest trend of improvement in male and female mortality at advanced ages accelerating in the most recent decades but still lagging behind those evident in England and Wales. Overall, we estimate that life expectancy for a 75 year old male was 7.0 years in 1951 rising to 9.1 years in 2002, higher than the official Irish Life Table estimates of 6.8 and 8.9 years respectively. Similar underestimates are found for female life expectancies. The shape of the curve at advanced ages is also different to that recorded in the official tables, with the rate of increase in mortality rates decelerating more markedly.

2. Graduation of the Irish Experience

Figure 1 highlights the need to graduate the Irish crude mortality rates found by the method of extinct generations in Part 1. Age heaping in the reported age of death at, say, 90 years, has created a crude mortality rate at age 90 years implausibly higher than the crude mortality rate of a 91 year old. The crude rates in England and Wales form a more regular curve, strictly increasing with age, than the crude Irish rates. Given the problems with the Irish data and the need to extrapolate the curve to very high ages, graduation by mathematical formula is considered preferable over more data-based techniques.

Source: Figure 13 from *Mortality in Ireland at Advanced Ages, 1950-2006: Part 1: Crude Rates*, Whelan (2009). The crude mortality rates for Ireland were calculated using the method of extinct generations and the mortality rates for males in England and Wales, are from Table 6 in Thatcher (1992).

Figure 1. Comparison of mortality rates for males ages 85 to 98 years, Ireland and England and Wales over decades 1950-60 and 1980-90

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

2.1 *Laws of Mortality*

Several mathematical formulae have been suggested that might parsimoniously capture how mortality rates change with advancing years of age. Olshansky & Carnes (1997) and Forfar (2004) give an overview of such so-called 'laws of mortality'. Often the laws express a relationship between age and the force of mortality at age x, generally denoted μ_x, which is the instantaneous rate of mortality at exact age x. We have the identity

$$q_x = 1 - \exp\left(-\int_x^{x+1} \mu_t \, \mathrm{d}t\right). \tag{1}$$

The approximation $q_x \cong 1 - \exp(-\mu_{x+0.5})$ or, rearranging,

$$\mu_{x+0.5} \cong -\ln(1 - q_x) \tag{2}$$

is very good, typically introducing a relative error of less than 0.05% in the approximation. We use (2) when estimating q_x given μ_x or vice versa.

A list of classic laws of mortality would include:

— Gompertz's Law (Gompertz (1825)): $\mu_x = \exp(\alpha + \beta x)$ (3)

— Makeham's Law (Makeham (1860)): $\mu_x = c + \exp(\alpha + \beta x)$ (4)

— Perks's Law (or Logistic Model) (Perks (1932)):

$$\mu_x = c + \frac{\exp(\alpha + \beta x)}{1 + \exp(\alpha + \rho + \beta x)} \tag{5}$$

— Perks's Law–Beard's Version (Beard (1964, 1971)):

$$\mu_x = \frac{\exp(\alpha + \beta x)}{1 + \exp(\alpha + \rho + \beta x)} \tag{6}$$

— Weibull's Law (Weibull (1951)): $\mu_x = ax^b$. (7)

Other models more recently suggested, based on goodness-of-fit for older ages over many mortality experiences, include

— Heligman–Pollard 1 (Heligman & Pollard (1980)): $q_x = \dfrac{GH^x}{1 + GH^x}$ (8)

— Heligman–Pollard 2: $q_x = \dfrac{GH^x}{1 + KGH^x}$ (9)

— Heligman–Pollard 3: $q_x = \dfrac{GH^{x^K}}{1 + GH^{x^K}}$ (10)

— Perks's Law–Kannisto Version (Thatcher *et al.* (1998)):

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

$$\mu_x = \frac{\exp(\alpha + \beta x)}{1 + \exp(\alpha + \beta x)}. \tag{11}$$

For brevity, we shall shorten Heligman–Pollard to HP and Perks's Law–Kannisto Version to simply Kannisto's model.

Mortality laws can be grouped by their limiting behaviour into three distinct classes, namely, into (i) those that assume that there is a fixed age limit human life, i.e. there exists an age ω such that $q_w = 1$, (ii) those that assume that there is no finite fixed age limit but mortality increases with advancing age to asymptotically reach unity, i.e. $q_x < 1$ for all x but $\lim_{x \to \infty} q_x = 1$, and (iii) those that assume the mortality peaks or reaches a plateau or asymptote below unity, i.e. there is a constant c, such for all x, $q_x \le c < 1$. It can easily be shown that none of the above laws are in the first group, that Gompertz's Law, Makeham's Law, Weibull's Law, HP 1 and HP 3 are all in the second group and the third group includes Perks's Law (including Beard's and Kannisto's version) and HP 2 (with $O < K < 1$ when it then limits to $1/K$).

2.2 *Fitting the Models*

It was decided to fit six models to the crude Irish mortality data at advanced ages determined by the method of extinct generations. Three model types were taken from the class of laws that assumed mortality increases with age asymptotically to unity — the Makeham, HP 1 and HP 3. The other three models chosen — the Logistic, Kannisto, and HP 2 — assume that the mortality rate tends to a plateau somewhat below unity. Figure 2 illustrates the different shape of the six curves, especially evident at advanced ages, when each curve was fitted to the reported Irish female mortality rates in 2001-2003 (CSO (2004)), by minimising the square of the relative error in the age range 75 years to 100 years.

Purely statistical techniques cannot be relied on to provide a satisfactory calibration of the models in the light of the data anomalies highlighted in *Mortality in Ireland at Advanced Ages, 1950-2006: Part 1: Crude Rates* (Whelan (2009)). Age heaping and age exaggeration at advanced ages, the extent of which varies with time, ensures that deviations from any fitted model, even the true underlying model, will have a significant non-random component. Indeed, the considerations in Section 3 of *Part 1* (Whelan (2009)) suggest that non-random deviations could be more significant than random deviations due to biases in the data, especially in the earlier calendar decades studied. Accordingly, the conditions are not satisfied to apply uncritically such statistical parameter estimation procedures as maximum likelihood or model selection criteria such as the Akaike Information Criterion. Extensive experiments show that all six models are rejected at conventional p-levels, with p-values less than 0.05% even when fit to the relatively sparse data sets

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

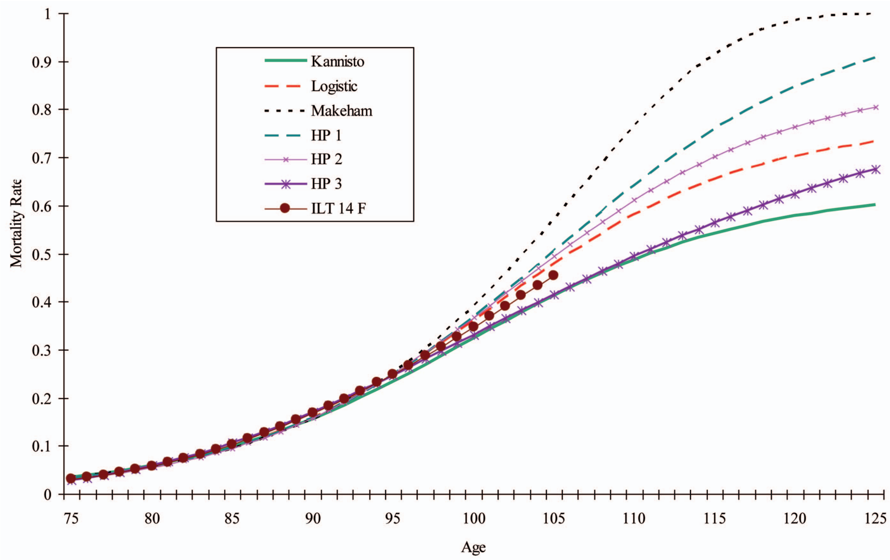


Figure 2. Several models fitted to ILT 14 F, by minimising squares of relative errors in age range 75 to 100 years

of mortality rates from a single calendar year or year of birth (see Section 3.2 later) and considerably lower than that when the data is aggregated over many calendar years. A more nuanced approach is required to model fitting and evaluation.

Experimentation by fitting the models over different age ranges and by different techniques revealed that no single model proved more satisfactory than any of the others: the model selected depended on the evaluation criteria. Two main approaches were settled on, which, though different, produce essentially the same results. In both approaches, the models parameters were estimated over a key part of the age range so the resultant curve adhered closely to the crude rates over that range, and the fitted curves were then extrapolated to higher and lower ages and the fit re-evaluated over these longer ranges using several different measures. Specifically, the models were fit in the age range 83 to 100 years. The parameterised model was then used to extrapolate mortality rates back to 75 years of age and over 100 years of age. The models were then assessed on

(1) Best fit in age range 83-100 years, as determined by weighted least squares.
(2) Best fit when extrapolated back to age 75 years, as determined by weighted least squares. This criterion was introduced to ensure that the fitted formula from age 83 would blend smoothly with mortality rates up

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

to this age. There are sufficient reliable data to enable Irish mortality rates up to ages the early or mid-80s to be estimated by other means.

(3) Best fit by unweighted least squares in age range 88 to 98 years. Weighted least squares gives considerably more weight to the younger ages in the age range. This criterion was introduced to test whether the fit was reasonable over the whole range 88 to 98 years.

The first method employed for parameter estimation in the fitted age range of 83 to 100 years was weighted least squares. Using a distance metric appears natural in this context, as we wish to monitor the closeness of the fitted model to the crude rates. Of course, estimation by maximum likelihood or minimum chi-square would have produced almost identical parameter estimates (see, for instance, Benjamin & Pollard (1980), p. 320). The second method was to estimate the parameters by minimising the weighted relative square error (as described in detail in the Appendix). The motivation for this alternative estimation procedure was that, as the level of mortality changes by a factor of four times over the fitted age range 83 to 100 years, it was desirable to ensure that the model would fit with equal proportionate closeness to all ages. In the event, the models fit by the two differing procedures were reassuringly very close to one another, as highlighted in Figure 3.

Model calibration was done on ten distinct data sets. Male and female mortality was modelled separately on both a calendar year and cohort basis. Deaths and exposed to risk were aggregated over the 11 calendar years 1950-60, 1960-70, and 1970-80, and also related to cohorts by year of birth in 11 year ranges 1875-1885 and 1885-1895. We set out the results of the modelling exercise when the parameters were estimated by weighted least squares in the next subsection and, in Appendix A, show the results when the parameters were estimated by minimising the weighted relative square error.

Of course, if one of the formulae captures the true underlying mortality curve that applied in age range from 75 years onwards, and the crude mortality rates were only subject to random fluctuations, then that model should perform well on all the tests. However, age heaping and other data anomalies ensure that no model is acceptable using purely statistical criteria of goodness-of-fit.

Figure 4 graphs the fitted models based on the aggregated deaths and exposed to risk for males born in years 1885 to 1995 inclusive, determined by the method of extinct generations.

2.3 *Evaluation of Calibrated Models*

Table 1a summarises the results when models were fitted to male mortality rates over the age range 83 years to 100 years by weighted least squares and evaluated on criteria (1)-(3) described in the previous subsection. Each numeric entry in the table gives an index of the goodness-of-fit for that model type (column heading), when fitted to the crude mortality rates

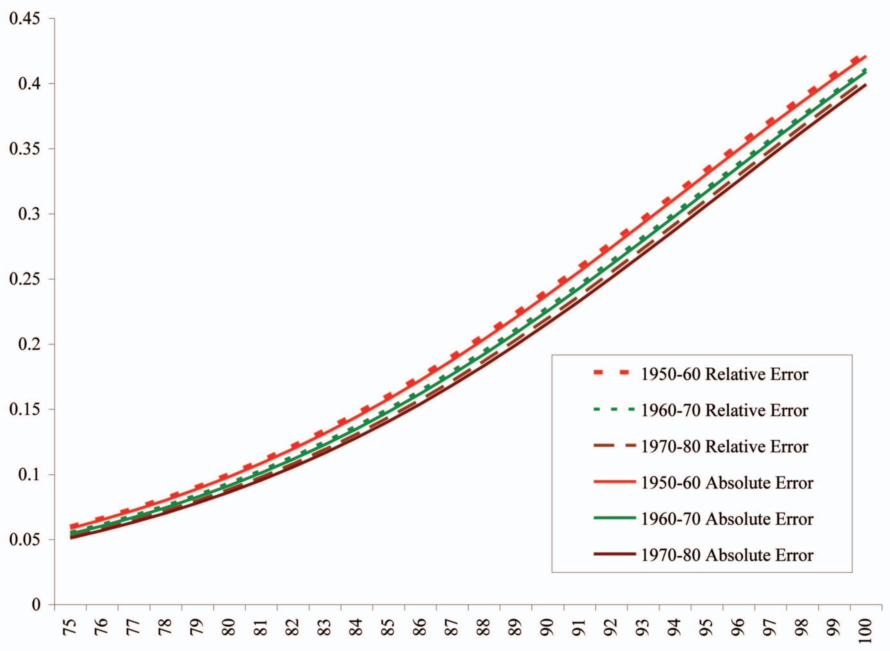


Figure 3. Kannisto models fit to Irish female crude mortality rates (by method of extinct generations), 1950-60, 1960-70, and 1970-80 using weighted least squares, relative and absolute error

over each of the five distinct periods (subgroup headings in first column), using the fit evaluation in row heading. Values in the table have been rebased to aid comparability across the two distinct measures employed (weighted least squares and unweighted least squares).

All models are emphatically rejected on a statistical basis, due to the same reason: the underlying crude mortality rates are too uneven to be fitted by a smooth curve. This is due to underlying data anomalies discussed earlier. However the table allows us to identify the closest fitting models under each criterion. Consider, for instance, the Makeham curve. Table 1a reports that the in-sample fit is one of the worst — being the worse fit in 3 of the 5 datasets and second worst fit (to the Kannisto) in the other two cases. However, in all cases, the margin of difference is not that material. The same calibrated Makeham curve, when the fit is considered over the longer range 75 to 100 years, gives the worse of all fits in each dataset and a fit materially looser than the Kannisto and HP 1, HP 2, and HP 3. We note, though, that the fitted Makeham, when evaluated in closeness of fit at the upper of the

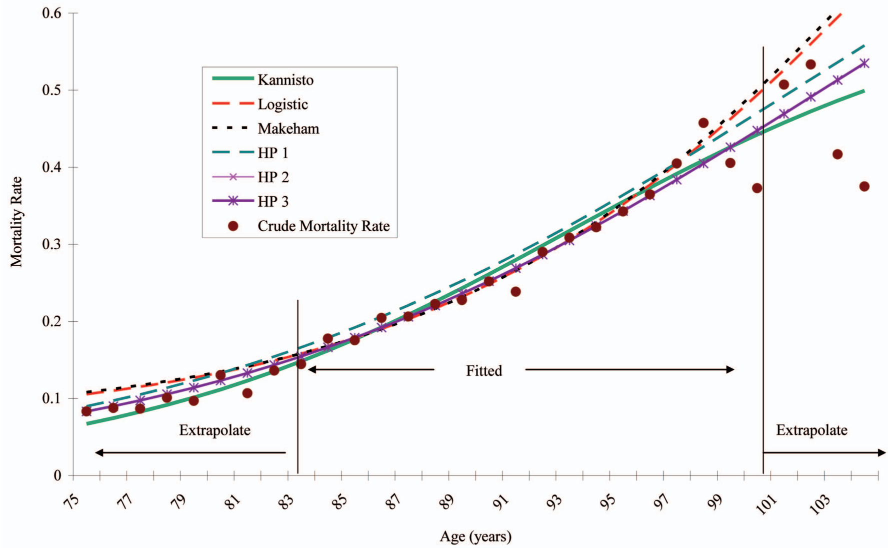


Figure 4. Mortality laws fit to crude mortality rates (by method of extinct generations), Irish male cohorts born, 1885-1995

fitted range by unweighted least squares, gives one of the best fits — either the best or second only to the Logistic.

The results in Table 1a produce a muddled picture. The in-sample fit gives little discrimination between the models, especially considering the data anomalies. The Makeham and Logistic laws can, perhaps, be ruled out as these models do not dove-tail nicely with mortality rates at younger ages outside the fitted range. Each of the other four models have strengths. The same remarks hold true for females as shown in Table 1b.

The probabilistic model underlying parameter estimation — whether weighted least squares, maximum likelihood or minimum chi-squared — tend to give considerably higher weightings to ages at the beginning of the age range fit, as the data count is higher at those ages. In the fitting procedure adopted, three-quarter of the weights applied to ages 83 to 87 years inclusive when fit over age range 83 to 100 years. This entails that the calibrated models might not be satisfactory if used at higher ages. Accordingly, we check the overall reasonableness of the models, especially at higher ages, by less formal procedures. Tables 2a and 2b show the life expectancy at ages 75, 85 and 95 years based on the crude mortality rates calculated from the data for males and females respectively, and the percentage deviation when life expectancies were calculated from the fitted models. Tables 2a and 2b also show the estimated mortality rate at age 100 years for each sex.

Table 1a. Evaluation of models, Irish males, aggregated over periods, by year of death and year of birth

	Kannisto	Logistic	Makeham	HP 1	HP 2	HP 3
Year of death: 1970-80						
Weighted least squares, ages 83-100	83	62	63	62	62	62
Weighted least squares, ages 75-100	513	1,267	1,388	269	238	276
(Unweighted) least squares, ages 88-98	76	29	31	64	58	65
Year of death: 1960-70						
Weighted least squares, ages 83-100	110	116	118	103	113	103
Weighted least squares, ages 75-100	451	1,577	1,755	441	368	445
(Unweighted) least squares, ages 88-98	136	84	82	116	122	116
Year of death: 1950-1960						
Weighted least squares, ages 83-100	162	155	156	151	152	151
Weighted least squares, ages 75-100	499	1522	1683	600	589	603
(Unweighted) least squares, ages 88-98	406	253	245	358	355	359
Cohort, born 1875-1885						
Weighted least squares, ages 83-100	97	100	101	92	92	92
Weighted least squares, ages 75-100	467	1,393	1,558	435	442	447
(Unweighted) least squares, ages 88-98	85	35	33	64	65	64
Cohort, born 1885-1895						
Weighted least squares, ages 83-100	64	65	67	56	56	56
Weighted least squares, ages 75-100	362	1,295	1,462	329	335	336
(Unweighted) least squares, ages 88-98	47	18	17	35	35	

Note: Values above have been rebased to aid comparability across the two distinct measures employed (weighted least squares and unweighted least squares). That is, the weighted least measure (whatever the age range fitted) has been multiplied by the same scaling constant, and the unweighted least square measure has similarly been rescaled. Of course the lower the value the better the fit.

Tables 2a and 2b highlight the unacceptable fit of the Makeham and Logistic models when extended back to 75 years of age, as the resultant implied life expectancies understate the life expectancy calculated directly from the data by about 10%. The HP 1 also shows an unacceptably poor fit to life expectancies at ages 75 and 85 years, again giving a significant underestimate. We note that the Kannisto model most closely reproduces the life expectancies at age 75 years estimated directly from the data.

2.4 *Selection of Kannisto Model*

The conclusion from the modelling exercise is that the Kannisto model tended to perform reasonably well when calibrated by different methods and

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

Table 1b. Evaluation of models, Irish females, aggregated over periods, by year of death and year of birth

	Kannisto	Logistic	Makeham	HP 1	HP 2	HP 3
1970-80						
Weighted least squares, ages 83-100	124	82	84	87	89	88
Weighted least squares, ages 75-100	408	2,111	2,572	721	795	771
(Unweighted) least squares, ages 88-98	71	38	36	61	69	65
1960-70						
Weighted least squares, ages 83-100	171	128	128	129	129	130
Weighted least squares, ages 75-100	558	1,594	1,915	723	754	686
(Unweighted) least squares, ages 88-98	130	81	78	123	127	119
1950-1960						
Weighted least squares, ages 83-100	274	220	219	239	238	238
Weighted least squares, ages 75-100	888	1,597	1,401	667	691	660
(Unweighted) least squares, ages 88-98	308	214	182	301	308	291
Cohort, born 1875-1885						
Weighted least squares, ages 83-100	114	100	112	102	102	101
Weighted least squares, ages 75-100	570	1,403	1,166	547	542	578
(Unweighted) least squares, ages 88-98	81	43	49	65	64	70
Cohort, born 1885-1895						
Weighted least squares, ages 83-100	80	76	88	70	70	70
Weighted least squares, ages 75-100	675	1,956	1,645	655	648	647
(Unweighted) least squares, ages 88-98	46	10	18	19	19	19

Note: Values above have been rebased to aid comparability across the two distinct measures employed (weighted least squares and unweighted least squares).

when extrapolated to lower ages than the range fit. Each of the other models was found not to have the same all-round robustness of the Kannisto model. The principle of parsimony also favours the two-parameter Kannisto model, as all the other fitted models except HP 1 have more parameters. It would appear that the general shape of the Kannisto curve better approximates the curve of mortality rates at advanced ages.

This general conclusion is supported by Thatcher *et al.* (1998) who conclude from modelling their considerable dataset of over 32 million deaths at advanced ages over the period 1960-1990 that "the logistic model and its Kannisto approximation are the best of the original six models". The six models they used were Gompertz's Law, Makeham's Law, Perk's Law, the Kannisto model, Weibull's model and HP 1 and the criteria used were primarily statistical goodness-of-fit tests given the quality of the underlying data. The study confirmed the growing consensus that the Gompertz,

Table 2a. Estimates of life expectancies, various ages, based on fitted models, Irish males, by year of death and year of birth

	Based on data*	Percentage deviation from column (2) when life expectancies calculated from the fitted models					
	(2)	Kannisto	Logistic	Makeham	HP 1	HP 2	HP 3
Based on period 1970-80							
Life expectancy at age 75	7.38	4.4%	−10.6%	−11.2%	−7.9%	−1.7%	−2.7%
at age 85	4.00	−1.4%	0.6%	0.7%	−6.1%	−0.3%	−0.1%
at age 95	2.07	2.1%	−1.3%	−0.5%	−1.7%	2.9%	4.5%
q_{100}	0.499	0.443	0.505	0.498	0.471	0.456	0.449
Based on period 1960-70							
Life expectancy at age 75	7.40	2.3%	−11.7%	−12.5%	−9.1%	−2.0%	−3.8%
at age 85	3.89	−0.5%	1.3%	1.3%	−5.5%	2.4%	0.5
at age 95	2.11	−0.9%	−4.5%	−5.3%	−6.1%	1.3%	−0.3%
q_{100}	0.446	0.447	0.507	0.514	0.484	0.456	0.462
Based on period 1950-60							
Life expectancy at age 75	7.20	0.7%	−10.4%	−11.2%	−9.4%	−4.1%	−4.2%
at age 85	3.85	−1.2%	0.3%	0.4%	−6.1%	−0.3%	−0.2%
at age 95	1.92	10.3%	2.4%	1.5%	3.6%	9.5%	9.8%
q_{100}	0.572	0.440	0.517	0.524	0.479	0.459	0.458
Cohort, born 1875-1885							
Life expectancy at age 75	7.31	1.7%	−10.2%	−11.0%	−8.1%	−2.9%	−3.0%
at age 85	3.92	−0.3%	1.0%	1.1%	−5.5%	0.5%	0.4%
at age 95	2.06	4.7%	−1.8%	−2.6%	−2.4%	3.8%	3.7%
q_{100}	0.424	0.434	0.504	0.511	0.476	0.453	0.454
Cohort, born 1885-1895							
Life expectancy at age 75	7.44	2.1%	−10.7%	−11.5%	−8.3%	−3.2%	−3.2%
at age 85	4.00	−0.8%	0.8%	0.9%	−5.9%	0.1%	0.1%
at age 95	2.13	2.1%	−2.6%	−3.3%	−3.8%	2.2%	2.2%
q_{100}	0.428	0.433	0.494	0.500	0.470	0.448	0.448

* q_{100} from the crude data is estimated as average of rates q_{99}, q_{100} and q_{101} due to the very uneven development of the crude rates at these ages. Typically the crude rate at age 99 years was materially higher than at age 100.

Makeham, Weibull and Heligman–Pollard 1 give a relatively poor fit and all tend to predict mortality rates far too high above age 100 years when fit to crude mortality rates in age range 80-98 and extrapolated.[1] Accordingly,

[1] Weibull's model could be made to give a good fit within the range of ages 85 to 105 but 'gives some highly dubious extrapolations' whether below age 85 years or at very high ages (110 to 120 years).

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

Table 2b. Estimates of life expectancies, various ages, based on fitted models, Irish females, by year of death and year of birth

	Based on data*	Percentage deviation from column (2) when life expectancies calculated from the fitted models					
	(2)	Kannisto	Logistic	Makeham	HP 1	HP 2	HP 3
Based on period 1970-80							
Life expectancy at age 75	9.02	0.5%	−11.0%	−12.5%	−10.1%	−5.4%	−5.3%
at age 85	4.77	−1.3%	0.3%	0.4%	−6.3%	0.2%	−0.1%
at age 95	2.40	2.3%	1.0%	0.9%	−1.0%	7.0%	6.2%
q_{100}	0.419	0.399	0.434	0.437	0.419	0.391	0.394
Based on period 1960-70							
Life expectancy at age 75	8.63	1.5%	−9.6%	−10.9%	−9.8%	−5.0%	−4.4%
at age 85	4.60	−1.4%	0.3%	0.4%	−6.1%	−0.1%	−0.2%
at age 95	2.28	4.1%	1.6%	1.1%	1.8%	8.8%	7.5%
q_{100}	0.439	0.409	0.452	0.458	0.426	0.402	0.408
Based on period 1950-60							
Life expectancy at age 75	8.11	3.3%	−9.1%	−8.4%	−7.6%	−2.7%	−2.1%
at age 85	4.40	−1.9%	1.0%	−0.2%	−6.4%	−0.5%	−0.7%
at age 95	1.98	14.8%	13.6%	9.0%	10.7%	18.2%	16.8%
q_{100}	0.594	0.421	0.462	0.485	0.446	0.422	0.428
Cohort, born 1875-1885							
Life expectancy at age 75	8.35	0.8%	−8.9%	−7.6%	−7.1%	−1.8%	−2.6%
at age 85	4.64	−0.8%	0.1%	−0.2%	−6.2%	−0.4%	−0.3%
at age 95	2.42	5.7%	−1.4%	−4.3%	−2.0%	3.9%	5.4%
q_{100}	0.403	0.380	0.439	0.456	0.416	0.396	0.390
Cohort, born 1885-1895							
Life expectancy at age 75	8.88	−4.5%	−11.2%	−9.8%	−9.5%	−4.5%	−4.5%
at age 85	4.86	0.5%	0.6%	0.5%	−5.9%	−0.2%	−0.2%
at age 95	2.59	9.4%	−2.0%	−4.5%	−2.9%	2.9%	3.0%
q_{100}	0.377	0.343	0.413	0.427	0.394	0.374	0.374

* q_{100} from the crude data is estimated as average of rates q_{99}, q_{100} and q_{101} due to the very uneven development of the crude rates at these ages. Typically the crude rate at age 99 years was materially higher than age 100.

their analysis settled on Perks's Law, and its special case, the Kannisto version as best capturing the pattern of late-life mortality. On the principle of parsimony, the conclusion is to fit the Kannisto model and, should the fit not be adequate, only then attempt the more general Perks's Law.

One of the strengths of the Kannisto model (and the more general Perks's Law) reported by Thatcher *et al.* (1998) is that they provide the best estimates of mortality rates when extrapolated above 98 years. Of course, it was not possible to form any reasonable estimate of Irish mortality at such

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

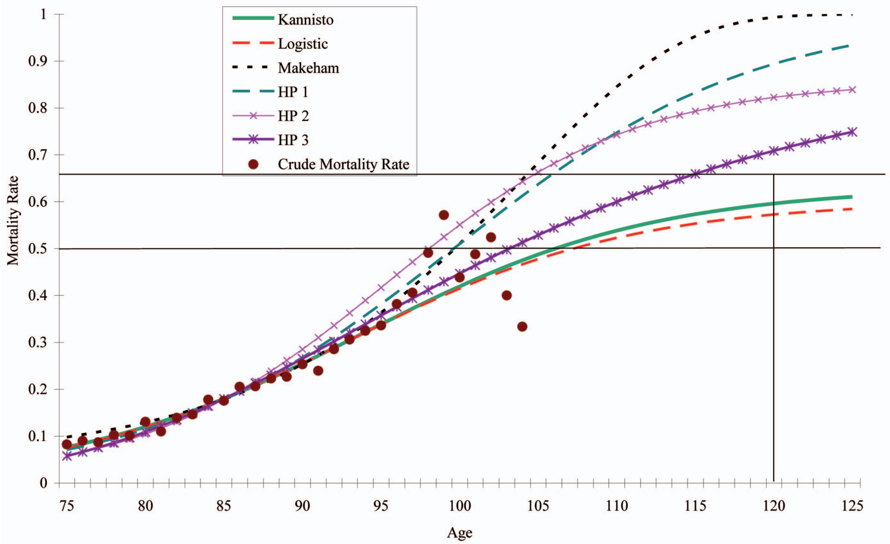


Figure 5. Mortality laws fit to crude mortality rates, Irish males, 1970-1980 and compared with crude rates (by method of extinct generations), by minimised weighted relative error in age range 83-100 years and extrapolated

high ages given the paucity of data. They estimate that q_{120} is between about 0.5 and 0.65 for both males and females (although perhaps the reported standard error is too low (Macdonald, 2001)). Extrapolations of the models fit to the Irish data produce estimates of q_{120} within this range for the Kannisto models, and sometimes for the logistic, but are always too high and outside the range for the other models. Figure 5 illustrates the typical pattern when the models fit to Irish mortality rates are extrapolated to very advanced ages.

The Kannisto model, as detailed in 2.1, has the form

$$\mu_x = \frac{\exp(\alpha + \beta x)}{1 + \exp(\alpha + \beta x)}.$$

Accordingly, under this model, the force of mortality is always increasing with age, with its rate of increase declining, that is:

$$\frac{d}{dx}\mu_x = \beta(\mu_x - \mu_x^2). \tag{12}$$

Materially, the model converges asymptotically, with $\lim_{x\to\infty}\mu_x = 1$, so that $\lim_{x\to\infty} q_x = 1 - e^{-1} \cong 0.632$. Accordingly, one can view the Kannisto

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

model as anchoring extremely advanced mortality rates at a level not inconsistent with large scale studies.

We conclude that Irish mortality at advanced ages is best modelled using the Kannisto model. Of course, over some time periods studied, the Kannisto model might be bettered by other models under certain criteria (for example the Logistic when fitted by minimising weighted relative error often outperforms its restricted version of the Kannisto on the goodness-of-fit criteria) but the requirement to apply a single model over many time periods so trends can be more easily identified favours the more robust and structured Kannisto model. Though it matters little (with estimated mortality rates over the age range 80-100 years always within 2% of each other), calibrating the model by minimising weighted relative error rather than weighted absolute error was considered marginally preferable. We shall minimise weighted relative error in subsequent model fitting.

2.5 *Level and Trends in Irish Mortality Rates at the Highest Ages, 1950-1980*

We now summarise the finding from graduating the mortality experience

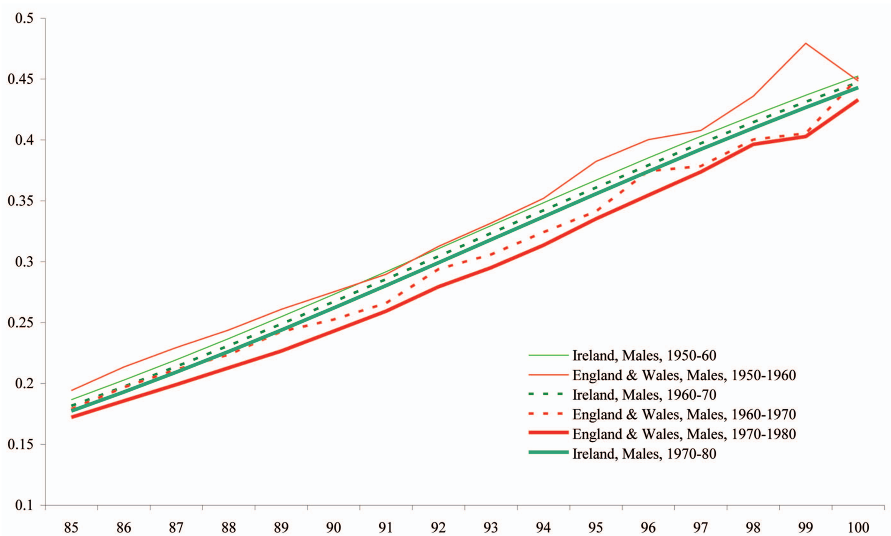


Figure 6. Mortality rates for males ages 85 to 100 years, Irish experience graduated, compared with crude rates in England and Wales, over decades 1950-60, 1960-70, and 1970-80

Note: Above can be compared with Figure 1, which graphs the crude Irish rates and comparison with those of England and Wales. Mortality Rates for Males in England and Wales from Thatcher (1992).

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

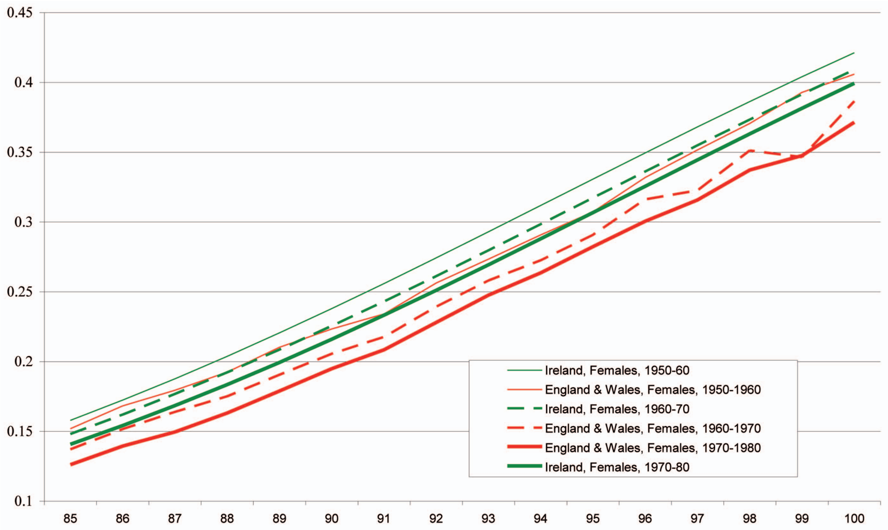


Figure 7. Mortality rates for females ages 85 to 100 years, Irish experience graduated, compared with crude rates in England and Wales, over decades 1950-60, 1960-70, and 1970-80

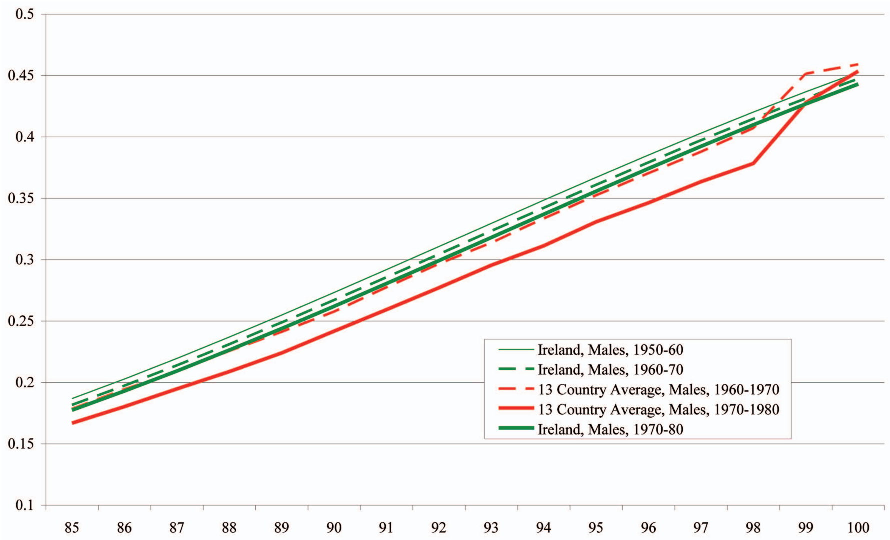
Note: Mortality Rates for Females in England and Wales from Thatcher (1992).

Table 3. $1000q_x$ from selected Irish Life Tables, males, ages 80 and over

Source	Period	Mortality rate at age x years, where x =				
		80	85	90	95	100
Males						
ILT 10	1981-82	122	182	257	346	424
Graduated experience	**1970-80**	**120**	**180**	**255**	**339**	**419**
ILT 8	1970-72	122	183	262	357	408
Graduated experience	**1960-70**	**116**	**181**	**264**	**356**	**442**
ILT 6	1960-62	125	199	296	418	571
Graduated experience	**1950-60**	**120**	**189**	**276**	**370**	**455**
ILT 5	1950-52	136	199	274	363	467
Females						
ILT 10	1981-82	87	139	209	294	394
Graduated experience	**1970-80**	**89**	**144**	**220**	**311**	**403**
ILT 8	1970-72	97	153	226	315	415
Graduated experience	**1960-70**	**96**	**151**	**228**	**320**	**411**
ILT 6	1960-62	104	164	243	339	454
Graduated experience	**1950-60**	**100**	**161**	**241**	**334**	**424**
ILT 5	1950-52	117	171	235	309	395

Source: From various Irish Life Tables by Central Statistics Office and author's calculations.

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press



Source: Mortality Rates for 13 Developed Countries from Thatcher *et al.* (1998).

Figure 8. Comparison of mortality rates for males ages 85 to 100 years, Ireland and 13 developed country average over decades 1950-1960, 1960-70 and 1970-80

using the Kannisto model, fitted to the crude mortality rates determined by the method of extinct generations by minimising the sum of the weighted relative errors in the range 83 to 100 years. We shall refer to this graduation simply as the graduated Irish experience.

Figures 6 and 7 well illustrates the small but clear trend of improvement in the Irish male and female mortality experience over the three decades 1950-80 and compares it with the experience in England and Wales over the same period.

Table 3 compares how our graduated experience compares with the official Irish Life Tables at selected ages over the different decades.

We may summarise our findings over the period 1950 to 1980 as follows:

(1) There has been a small but discernible trend of improvement in male and female mortality at advanced ages (of about 0.5% per year), with the rate of improvements declining with increasing age. Female mortality has been improving marginally faster than male mortality.

(2) Mortality at age 100 years has been declining and it is estimated to be 0.42 for Irish males and 0.40 for Irish females in the 1970s.

(3) Improvements at advanced ages in Ireland have not been as great as those in England and Wales over the same period. Mortality for males in

England in the 1950s was marginally higher than that for males in Ireland at advanced ages but in the 1970s was about 5% lower. Irish female mortality was some 7% higher than that of females in England and Wales in the 1950s but the gap has widened so that in the 1970s Irish female mortality at advanced ages is about 10% higher.

(4) Irish data is consistent with the hypothesis that there is no maximum lifespan and that mortality rates plateau at very advanced ages at a level below unity. The best fit model, when extrapolated to very advanced ages, tentatively suggests that the mortality rate is only 0.61 at age 125 years, (and almost identical for males and females) and the extrapolated mortality rate at this age is declining very slowly with the passage of time.

Finally, Irish mortality is at a higher level and showing more modest rates of decline that those in many developed countries as illustrated in Figure 8.

3. Extending the Method of Extinct Generations

3.1 *Methods to Estimate Exposed-to-risk when Generations are not yet Extinct*

Define E_x^y as the initial exposed to risk at age x in calendar year y corresponding to the deaths aged x last birthday in calendar year y given by d_x^y. The method of extinct generations gives:

$$E_x^y = \sum_{i=0}^{\omega} d_{x+i}^{y+i}. \tag{13}$$

To apply (13) in practice requires that by calendar year $y+\omega$, the cohort aged x in calendar year y have all died. Let us assume that everyone will die before, say, their 112th birthday. Our database comprises all deaths in Ireland, subdivided by age and sex in each calendar year from 1950 to 2006. Accordingly, the method of extinct generations allows us to estimate the exposed to risk and therefore the mortality rate at age 111 in 2006, at age 110 in 2005, . . . , at age 90 in 1985. We desire a way to extend the method of extinct generations to cohorts not yet extinct so we can estimate the mortality rate of, say, a 90 year old in more recent calendar years than 1985.

Let p (for 'present') be the most recent calendar year for which we have the number of deaths. The cohort may not be all dead by the end of calendar year p and so we need an estimate of the number alive at that time, i.e. the number of survivors in the cohort at the start of calendar year $p+1$. So (13) must be modified to:

$$E_x^y = \sum_{i=0}^{i=p-y} d_{x+i}^{y+i} + E_{x+(p+1-y)}^{p+1}. \tag{14}$$

The crude mortality rate at age x in calendar year y is then given by:

$$q_x^y = \frac{d_x^y}{E_x^y}. \tag{15}$$

Three methods have been proposed to date to extend the method of extinct generations. Two involve methods to estimate E_z^{p+1}, namely, the survivor ratio method (see, for instance, Thatcher, 1992) and the Das Gupta method (Das Gupta, 1990). Another method, proposed by Andreev (1999), known as the Mortality Decline method, estimates a log-linear age-specific decline in mortality from previous cohorts to estimate the survivor count of an unexpired cohort. Our earlier analysis suggests another approach, also based on extrapolating mortality rates, which we now describe.

We can estimate the number in each cohort still alive at the end of the period based on the assumption that the Kannisto model adequately models late-life mortality. This novel approach may be summarised by the following recursive procedure:

(1) fit a Kannisto mortality curve to the last extinct cohort and use this as an initial estimate for the mortality curve for those born one calendar year later. [Alternatively, in times of rapid mortality change, fit a Kannisto curve to each of the last n extinct cohorts and extrapolate the trend in the two fitted parameters to identify the Kannisto curve most likely to provide a reasonable fit to those born one year later.]

(2) Apply the Kannisto curve in (1) to estimate the exposed to risk $E_{x+(p+1-y)}^{p+1}$, via the formula,

$$\hat{E}_x^y = \frac{\sum_{i=0}^{i=p-y} D_{x+i}^{y+i}}{1 - {}_{p+1-y}\,p_x}. \tag{16}$$

Where the mortality function, ${}_{p+1-y}p_x$, is estimated from the Kannisto curve from (1). Hence, from (14), we have:

$$\hat{E}_{x+(p+1-y)}^{p+1} = \hat{E}_x^y - \sum_{i=0}^{i=p-y} D_{x+i}^{y+i}. \tag{17}$$

(3) Now, with an initial estimate of $E_{x+(p+1-y)}^{p+1}$, we can calculate the crude mortality rate for the birth cohort in the subsequent calendar year.

(4) Fit another Kannisto curve to the crude mortality rates so obtained, to update the best estimate of mortality curve. If there is a significant difference between the initial estimate and this update then repeat

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

procedure from (2) using the updated estimate. Stop the iterative procedure when there is an immaterial difference between two successive iterations.

3.2 *Evaluation of Different Methods to Extend the Method of Extinct Generations*

A study of the relative performance of the different extensions was made in Thatcher *et al.* (2002), which compares their relative performance in nine countries over a period of 35 years. The study evaluates the performance of the survivor ratio method, the Das Gupta method and the Mortality Decline method and concludes that errors from each method tend to underestimate the observed population at higher ages by the order of 5 to 15%. This is confirmed by Andreev (2004), which shows that the survivor method understates the population aged 90 years and over in England and Wales by 8.4% over the period 1980-1995 when compared with the count by the method of extinct generations, and errors of this magnitude are not unusual (see especially Table 1 therein).

However, we note from *Mortality in Ireland at Advanced Ages, 1950-2006: Part 1: Crude Rates*, Whelan (2009), (see Figure 8 and Sections 3.2 and 4.2 therein) that the census count, suitably adjusted, can approximate the required exposed-to-risk to within 0-10% (averaging at about 5%) in an Irish context. We summarise the findings of our reconciliation attempts in Section 4.2 of *Mortality in Ireland at Advanced Ages, 1950-2006: Part 1: Crude Rates*, Whelan (2009) in Table 4 and Figure 9. The table and graph show how close the census count, suitably adjusted, can approximate the required exposed-to-risk.

Figure 9 and Table 4 show that using the census data, suitably adjusted, is probably a more reliable way of estimating the exposed-to-risk in more recent times than the three proposed extensions to the method of extinct generations. The adjustments to the census data are key because, as discussed in Section 4.2 of *Part 1*, the rationale for using the method of extinct generations is that it achieves a closer correspondence between the death data and the exposed to risk — not, as formerly believed, because it corrects age mis-statements. Note that using census data tends to underestimate the exposed-to-risk by about 5%, no doubt largely due to an undercount of those at advanced ages. [The exception is the older age groups in 1951 which are perhaps attributable to residual age exaggeration for pension purposes.]

Thatcher *et al.* (2002) report that the three proposed methods produce considerably better estimates of the exposed-to-risk if the initial estimates are scaled so that they are made to match an independently estimated population count (say, at or above age 90 years). Such constrained methods tend to reduce the error to a 1-5% range, but tend to overstate the exposed-to-risk. They concluded that in all cases the survivor ratio method when constrained to match the official estimates of population at and over age 90

Table 4. Ratio of adjusted population count in censuses of 1951, 1961, 1971 and 1981 to count by method of extinct generations, males and females, various age groups

Age group	1951	1961	1971	1981
Males				
75 years and over	0.975	0.973	0.960	0.965
80 years and over	0.967	0.937	0.956	0.944
85 years and over	1.014	0.926	0.957	0.949
90 years and over	1.089	0.937	0.974	1.062
Females				
75 years and over	0.983	0.962	0.926	0.951
80 years and over	0.972	0.931	0.920	0.936
85 years and over	1.015	0.914	0.904	0.923
90 years and over	1.091	0.908	0.871	1.010

Source: See Figure 10 and discussion in Section 4.2 in *Mortality in Ireland at Advanced Ages, 1950-2006: Part 1: Crude Rates*, Whelan (2009).

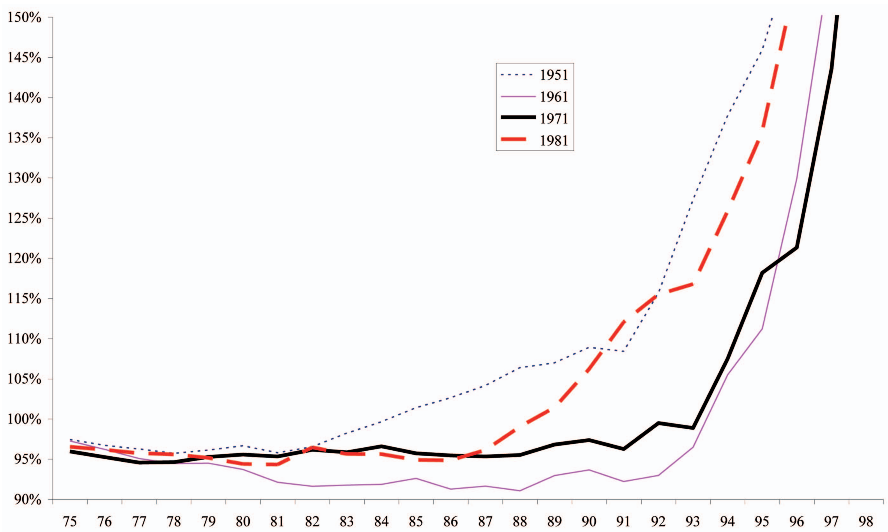


Figure 9. Ratio of adjusted population count in censuses of 1951, 1961, 1971, 1981 to count by method of extinct generations, males, various age groups at and over age point in graph

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

years was best. A more recent study, Andreev (2004), reports that a development of Das Gupta's method performs even better for larger populations and, in particular, does not exhibit a bias like the constrained survivor ratio. However, for smaller populations — even larger than that of Ireland — the variant of the Das Gupta method shows little or no improvement over the constrained survivor method.

We attempted to estimate the number surviving in each cohort based on the assumption that the Kannisto model adequately models late-life mortality, as described in the previous subsection. Figures 10a and 10b plot the two parameters that produce the best fitting Kannisto curve for those born in each of the years 1875 to 1895, for females and males respectively. The fitting procedure is the same as that described and applied earlier (see 2.2-2.5). The parametric form of the Kannisto model used for fitting purposes was $\mu_x = \dfrac{a\,e^{bx}}{1 + a(e^{bx} - 1)}$ to aid comparisons with models fit in Thatcher *et al.* (1998).

Figures 10a and 10b show that parameters *a* and *b* are dependent parameters, negatively correlated with each other, so that a fit can be found with an unusually high parameter *a* coupled with an unusually low *b* or vice versa. So the best fit parameters are very sensitive to the underlying data, with the possibility that small changes in the crude mortality rates can have a

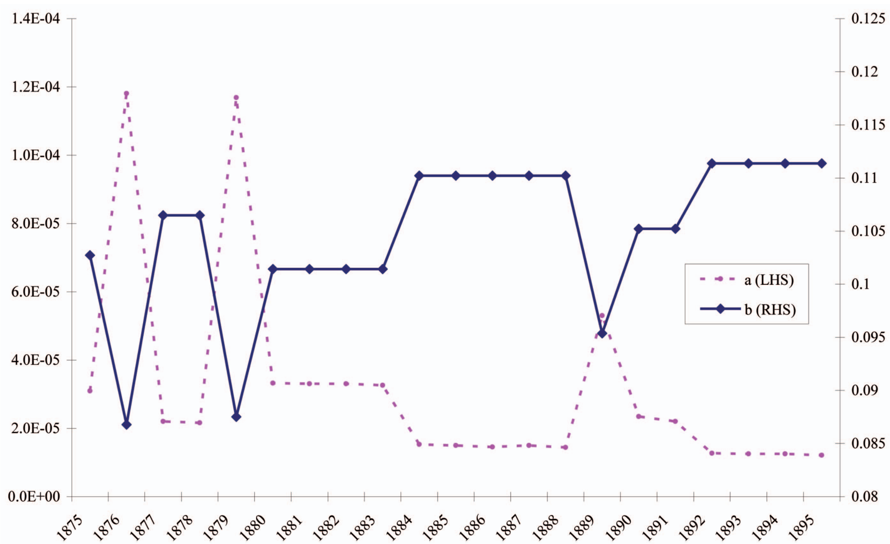


Figure 10a. Parameters of best fit Kannisto (weighted relative error), ages 83-100, Irish female cohorts born 1875-1895

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

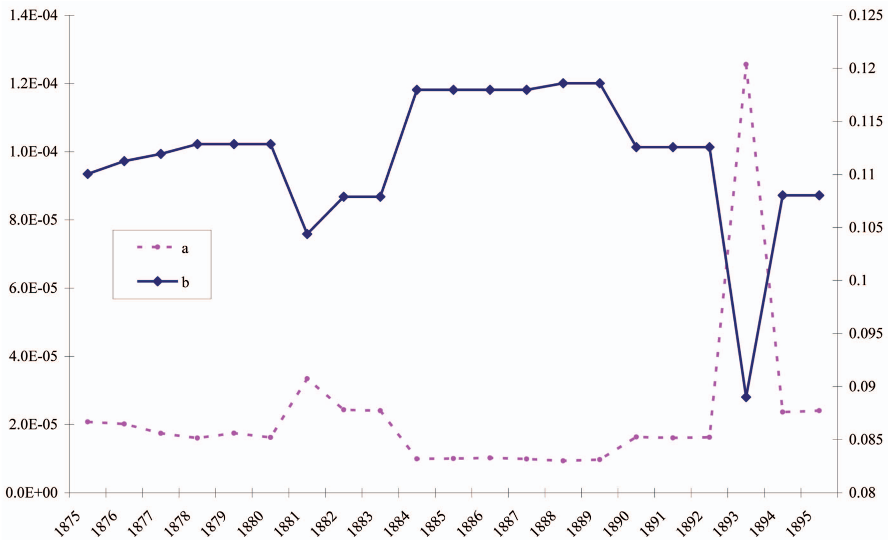


Figure 10b. Parameters of best fit Kannisto, ages 83-100, Irish male cohorts born 1875-1895

large impact on parameter estimates. For instance, take the fit to females born in 1879. The minimum chi-squared value of the best fit was 43.3 (which was close to the average across all fits), which, with 16 degrees of freedom has a p-value of 0.025%. However, taking values of a and b closer to the average observed over the 20 cohorts, with $a = 2.15\text{E-}05$ and $b = 0.1068$, the chi-squared value is 58.4 (which has an associated p-value of 0.000097%). Given the paucity of data points and the problems of age heaping identified with the data, the proposed method based on fitted Kannisto curves to estimate the numbers surviving in each cohort is simply not robust enough with the Irish data. [The analysis also shows the need to group Irish data over many calendar years or years of birth to reduce random error in the crude rates producing a rogue fit. These considerations, and the very gradual improvements with time identified, have informed the decision to group death data into 11 year groups.]

The conclusion from this subsection is that the most promising method of extending the method of extinct generations must, in some way, constrain the numbers estimated surviving at the end of the period by independent estimates. However, it appears a feature of census counts on the island of Ireland that they undercount the population at advanced ages.[2] From experiments with Irish data, we estimate that the desired survivor count can be estimated to within about 5% of the true number by suitably adjusting the census count, with a bias towards an under-estimate. This contrasts with the constrained survivor ratio, the best of the existing methods, which, when studied in other national datasets, tended to overstate the exposed-to-risk by, on average, between 1% and 5%.

3.3 *Estimating Irish Mortality at Advanced Ages in Recent Decades*

The survivor ratio method attempts to estimate E_{x+1}^{p+1} based on estimating the ratio $R_x^p(k)$, defined as

$$R_x^p(k) = \frac{E_{x+1}^{p+1}}{\sum_{i=0}^{k} d_{x-i}^{p-i}}. \tag{18}$$

A reasonable approximation to $R_x^p(k)$ might be to use the observed ratio the calendar year earlier, that is $R_x^{p-1}(k)$, and, applying it gives:

$$E_{x+1}^{p+1} \cong R_x^{p-1}\left(\sum_{i=0}^{k} d_{x-i}^{p-i}\right). \tag{19}$$

So (19) approximates E_{x+1}^{p+1} using information known before time $p+1$ and hence derives mortality rates at each age up to time p.

In fact, two variants of the survivor ratio method are employed in practice:

(i) Rather than take $R_x^p(k) \cong R_x^{p-1}(k)$, we can average the ratio over the m immediately preceding cohorts so as to create a more stable ratio, i.e.

$$R_x^p(k,m) \cong \frac{1}{m}\sum_{i=1}^{m} R_x^{p-m}(k). \tag{20}$$

(ii) In the case that mortality is believed to be changing over the period then we could put

$$R_x^p(k,m) \cong c \cdot R_x^{p-1}(k,m), \text{ for some constant } c, \text{ with } c > 1 \text{ in the case of mortality decline.} \tag{21}$$

Thatcher *et al.* (2002) suggests employing both (i) and (ii), by taking $m = k = 5$ and determining c by the constraint that the population estimated at 90 years of age and over at time $p+1$ be made to match independent estimates of the surviving population. This variant of the constrained survivor ratio method is now used in estimating mortality at the highest ages

[2] In the Republic of Ireland, the actual census count, unadjusted, is reported. This can be expected to always under-count the true population.

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

in both the Human Mortality Database and the Kannisto-Thatcher Database.

For the Irish data, we must take appropriate values of m and k to estimate $R_x^p(k, m)$, together with the age at which reconciliation to independent population estimates are made so c can be determined. For smaller populations, like that of Ireland, the estimate of $R_x^p(k, m)$ is subject to larger random fluctuations for any even k and m, suggesting that these parameter be increased for smaller populations. Simulations show that the coefficient of variation of $R_x^p(5, 5)$ is about 10% in the early to mid-90s year of age in populations the size of Ireland, rising rapidly with age. The crude mortality rates deduced from the constrained survivor method also depend on how the survivor estimates are scaled to the population estimate.

In contrast to the constrained survivor method, it would appear more straightforward to simply estimate the surviving population at the end of the period. Our previous investigations suggest that, by suitably adjusting the census count, we can estimate the surviving population to within about 5% of the true surviving count at ages over 83 years. Indeed, the 5% is more a bias towards an under-count (no doubt largely due to an undercount in the census itself), rather than a varying amount. It follows that we can achieve a reasonable extension to the method of extinct generations by simply (i) adjusting the census count as detailed in Section 4.2 of *Part 1* so that it better corresponds to the surviving numbers, (ii) estimating the resultant crude mortality rates, and, (iii) fitting a suitable curve to the crude rates as outlined in Section 2.

Ireland's most recent census was in April 2006. Accordingly, the most recent independent population estimate that can be used to constrain survivor estimates is at the start of 2006. We directly estimated the number of survivors of each cohort at the start of 2006 by adjusting the census count at each age as detailed in Section 4.2 of *Part 1* and assuming that the resultant adjusted census number represented alternatively (a) 100% of the survivors or, alternatively, (b) 95% of the survivors (so multiplied by 105%). We then calculated the crude mortality rates and fitted the six mortality curves to the crude rates as outlined in Section 2. Full details of the goodness-of-fit statistics of the models, in the same manner as earlier fits, are given in the Appendix, Tables A.5 and A.6. Once again, the Kannisto model produces acceptable fits across the different datasets.

4. Conclusion

Table 5 summarises the mortality rate of males and females in Ireland at advanced ages over the five decades from the 1950s to the 1990s. We note that there is an insignificant difference over the 1990s in the two alternative approaches in estimating the survivors at the end of the period. The best fitting Kannisto curve is shown, from which the mortality rates are derived.

Table 5. Irish mortality at advanced ages, estimated by Kannisto model

Period	Fitted parameter values of Kannisto		$1000q_x$ where $x =$				
	$a \cdot 10^5$	b	80	85	90	95	100
Males							
1990-2000 (105%E)	2.18	0.1074	104	160	234	318	402
1990-2000 (100%E)	2.19	0.1074	105	161	235	319	404
1980-90	2.99	0.1049	115	173	247	331	412
1970-80	3.15	0.105	120	180	255	339	419
1960-70	1.36	0.115	116	181	264	356	442
1950-60	1.12	0.118	120	189	276	370	455
Females							
1990-2000 (105%E)	0.385	0.1225	66	113	182	272	370
1990-2000 (100%E)	0.403	0.1221	67	114	183	273	370
1980-90	1.29	0.110	80	127	193	275	363
1970-80	0.806	0.117	89	144	220	311	403
1960-70	0.888	0.117	96	151	228	320	411
1950-60	0.932	0.117	100	161	241	334	424

Note: The parametric form of the Kannisto model used for fitting purposes was $\mu_x = \dfrac{a\,e^{bx}}{1 + a(e^{bx} - 1)}$.

Full details of the different models fit and the statistics of the goodness-of-fit are given in Table A.5 and Table A.6 in Appendix A.

We may summarise and update our earlier findings to cover the entire period 1950 to 2000 as follows:

(1) The trend of improvement in male and female mortality at advanced ages was very modest in early decades but has accelerated in the most recent decades.

(2) The rate of improvement declines with increasing age.

(3) Female mortality has been improving marginally faster than male mortality up to ages in the early nineties. At very advanced ages no or little improvement is discernible.

(4) The Kannisto model gives a reasonable fit to Irish mortality at advanced ages.

(5) Irish data is consistent with the hypothesis that there is no maximum lifespan and that mortality rates plateau at very advanced ages at a level below unity. The best fit model, when extrapolated to very advanced ages, tentatively suggests that the mortality rate is only 0.61 at age 125 years (and almost identical for males and females), and the extrapolated mortality rate at this age is declining very slowly with the passage of time.

(6) Mortality rates at most advanced ages are marginally lower than

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

Table 6. Irish mortality at advanced ages, 2001-03, estimated by Kannisto model

Period	Fitted parameter values of Kannisto		$1000q_x$ where $x =$				
	$a \cdot 10^5$	b	80	85	90	95	100
Males							
2001-03 (105%E)	1.24	0.1118	88	139	210	296	385
2001-03 (100%E)	1.46	0.1104	91	143	214	299	387
ILT 14 (2001-03)	—	—	89	145	220	313	424
Females							
2001-03 (105%E)	0.283	0.1243	57	100	164	251	350
2001-03 (100%E)	0.288	0.1245	59	103	168	257	356
ILT 14 (2001-03)	—	—	58	104	167	249	349

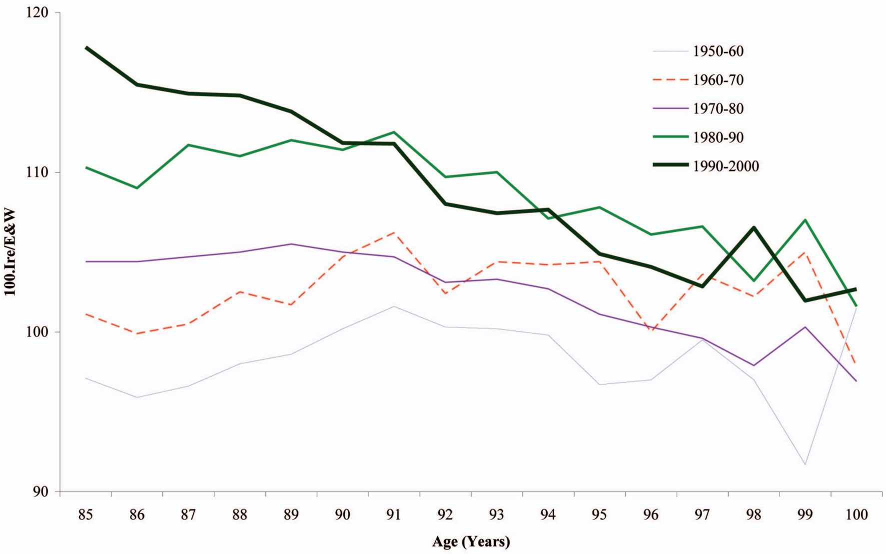
Note: The parametric form of the Kannisto model used for fitting purposes was $\mu_x = \dfrac{a\,e^{bx}}{1 + a(e^{bx} - 1)}$.

recorded in the Irish Life Tables, so life expectancies at advanced ages are slightly higher than previously believed. The more recent Irish Life Tables are more accurate than earlier tables. Table 6 contrasts our estimate of mortality at advanced ages in Ireland in 2001-2003 with Irish Life Table 14, CSO (2004). Details of the fit, and alternative models, are given in Tables A.7 and A.8 in Appendix A.

(7) Finally, the improvement in Irish rates has lagged behind that observed in England and Wales, as outlined in Figure 11. Mortality rates in the 1950s were lower in Ireland than England and Wales in the 1950s but are now higher.

Acknowledgements

Research funding for this project was kindly made available by a Government of Ireland Research Fellowship from the Irish Research Council for the Humanities and Social Sciences. The author acknowledges with thanks the Central Statistics Office, especially Helen Cahill, Joseph Keating and Aidan Punch, and the Office of the President of Ireland, especially Linda Hand, for kindly supplying data. I thank attendees at a statistics seminar in the UCD School of Mathematical Sciences in autumn 2008 for comments on some of the modelling approaches. I also thank two anonymous referees who made suggestions that improved the final paper. The paper is dedicated to mum, rest in peace, who did not make it to one of the advanced ages studied.



Sources: For England and Wales, ungraduated rates in Thatcher (1992) and, for period 1994-1998 (which approximates 1990-2000), from Gallop & Macdonald (2005). Irish rates as graduated by author.

Figure 11. Ratio of Irish male mortality to that of England and Wales, each decade 1950-2000.

References

Andreev, K.F. (1999). *Demographic surfaces: estimation, assessment and presentation, with application to Danish mortality*. Unpublished PhD dissertation, University of South Denmark.

Andreev, K.F. (2004). A method for estimating size of population aged 90 and over with applications to the 2000 U.S. Census Data. *Demographic Research*, **11**(9), 235-262.

Andreev, K., Jdanov, D., Soroko, E. & Shkolnikov, V. (2003). *Methodology: Kannisto–Thatcher database on old age mortality*. Max Planck Institute for Demographic Research, Rostock, Germany. [Online].

Beard, R.E. (1964). Some observations on stochastic processes with particular reference to mortality studies. *International Congress of Actuaries*, **3**, 463-477.

Beard, R.E. (1971). Some aspects of theories of mortality, cause of death analysis, forecasting and stochastic processes. In (W. Brass, ed.) *Biological Aspects of Demography*, Taylor & Francis Ltd, London.

Benjamin, B. & Pollard, J.H. (1980). *The analysis of mortality and other actuarial statistics*. Heinemann, London, 2nd edition.

CSO (2004). *Irish Life Tables no. 14, 2001-2003*. Central Statistics Office. Published by the Stationery Office, Dublin.

Das Gupta, P. (1990). *Reconstruction of the age distribution of the extreme aged in the 1980 census by the method of extinct generations*. Washington, DC 20233: Population Division of US Bureau of the Census, 1990.

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

Forfar, D.O. (2004). *Mortality laws*. In (J.F. Teugels & B. Sundt, eds.) *Encyclopedia of Actuarial Science*, **2**, 1139-1145.

Gallop, A.P. & Macdonald, A.S. (2005). *Mortality at advanced ages in the United Kingdom.* Paper presented to Society of Actuaries Symposium, *Living to 100 and beyond.* January 12-14, Florida.

Gompertz, B. (1825). On the nature of the function of the law of human mortality and on a new mode of determining the value of life contingencies. *Phil. Transactions of Royal Society*, **115**, 513-585.

Heligman, L. & Pollard, J.H. (1980). The age pattern of mortality. *Journal of the Institute of Actuaries*, **107**, 49-80.

Macdonald, A.S. (2001). Book Review: The Force of Mortality at Ages 80 to 120 by A.R. Thatcher, V. Kannisto and J.W. Vaupel (1998). *The Statistician*, **50**(1), 115.

Makeham, W.M. (1860). On the law of mortality. *Journal of the Institute of Actuaries*, **XIII**, 325-358.

Olshansky, S.J. & Carnes, B.A. (1997). Ever since Gompertz. *Demography*, **34**, 1, 1-15.

Perks, W. (1932). On some experiments on the graduation of mortality statistics. *Journal of the Institute of Actuaries*, **63**, 12-40.

Thatcher, A.R. (1992). Trends in numbers and mortality at high ages in England and Wales. *Population Studies*, **46**, 411-426.

Thatcher, A.R., Kannisto, V. & Vaupel, J.W. (1998). *The force of mortality at ages 80 to 120*. Odense University Press, Odense, Denmark.

Thatcher, A.R., Kannisto, V. & Andreev, K. (2002). The survivor ratio method for estimating numbers at high ages. *Demographic Research*, **6**(1), 1-18.

Weibull, W. (1951). A statistical distribution function of wide applicability. *Journal of Applied Mechanics*, **18**, 293-297.

Whelan, S.F. (2009). Mortality in Ireland at advanced ages, 1950-2006: Part 1: crude rates. *Annals of Actuarial Science*, **4**, 33-66.

APPENDIX A

EVALUATION OF MORTALITY MODELS FIT BY WEIGHTED RELATIVE SQUARE ERROR

We experimented with other methods of fitting and evaluating the models. One approach used was to fit the models by minimising the weighted *relative* square error rather than the absolute weighted square error used previously, that is to replace $(\hat{q}_x - q_x)^2$ with $\left(\frac{\hat{q}_x - q_x}{q_x}\right)^2$ in the sum to be minimised, so the quantity to be minimised was:

$$\sum_x \frac{E_x}{q_x(1-q_x)}\left(\frac{\hat{q}_x - q_x}{q_x}\right)^2. \tag{A.1}$$

The motivation for this change was that the level of mortality changes by a factor of four times over the fitted age range 83 to 100 and we wished to ensure that the model would fit with equal proportionate closeness to the lower ages as the higher ages. For computational convenience, we actually minimised the following:

$$\sum_x \frac{E_x}{\hat{q}_x(1-\hat{q}_x)}\left(\frac{\hat{q}_x - q_x}{q_x}\right)^2. \tag{A.2}$$

As before, we fitted the models in the age range 83 years to 100 years (inclusive), extrapolating the model back to 75 years, and evaluating the models on (i) weighted relative error in fitted range 83 to 100 years, (ii) weighted relative error in range 75 to 100 years, (iii) unweighted error in range 88 to 98 years, (iv) comparing life expectancies at ages 75, 85 and 95 years calculated from the fitted models with that calculated directly from the underlying crude mortality rates, and (v) comparing mortality rate estimates at age 100 years. The models fitted by this latter approach tended to better approximate the estimated life expectancy at age 75 years, but Makeham's law together with HP 2 and HP 3 produce the worse comparisons. The fitted Kannisto model tends to more closely approximate the crude life expectancies. The results of this alternative model fitting approach are set out in the Tables A.1-8. This procedure produces almost identical parameter estimates to the weighted least squares approach used in the main body of the paper and, accordingly, leads to the identification of the Kannisto model as preferable over the other models.

Table A.1. Evaluation of models, Irish males, aggregated over periods, by year of death and year of birth. Relative square error

	Kannisto	Logistic	Makeham	HP 1	HP 2	HP 3
1970-80						
Weighted least squares, ages 83-100	45	45	50	73	115	53
Weighted least squares, ages 75-100	94	100	239	202	740	636
(Unweighted) least squares, ages 88-98	101	108	36	38	239	83
1960-70						
Weighted least squares, ages 83-100	80	80	96	104	119	76
Weighted least squares, ages 75-100	187	155	251	210	608	533
(Unweighted) least squares, ages 88-98	138	134	102	94	264	138
1950-60						
Weighted least squares, ages 83-100	101	99	104	123	135	99
Weighted least squares, ages 75-100	243	191	257	249	647	570
(Unweighted) least squares, ages 88-98	385	369	275	276	457	383
Cohort, born 1875-1885						
Weighted least squares, ages 83-100	74	71	84	99	129	71
Weighted least squares, ages 75-100	261	183	212	265	828	686
(Unweighted) least squares, ages 88-98	85	76	71	41	256	84
Cohort, born 1885-1895						
Weighted least squares, ages 83-100	54	46	60	76	118	53
Weighted least squares, ages 75-100	223	118	191	211	764	630
(Unweighted) least squares, ages 88-98	54	67	59	23	256	54

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

Table A.2. Estimates of life expectancies, various ages, based on models, Irish males, aggregated over periods, by year of death and year of birth. Relative square error

	Based on data*	Kannisto	Logistic	Makeham	HP 1	HP 2	HP 3
1970-80							
Life expectancy at age 75	7.38	−0.1%	0.4%	−7.4%	2.6%	8.3%	7.5%
at age 85	4.00	−0.5%	−0.6%	−0.2%	−2.8%	−5.7%	−2.6%
at age 95	2.07	8.1%	8.8%	−3.6%	−7.6%	−16.3%	1.6%
q_{100}	0.499	0.419	0.415	0.510	0.510	0.551	0.447
1960-70							
Life expectancy at age 75	7.40	1.8%	0.2%	−6.1%	1.1%	6.8%	5.9%
at age 85	3.89	0.1%	−0.2%	0.5%	−1.6%	−4.4%	−1.2%
at age 95	2.11	0.6%	0.1%	−10.8%	−10.3%	−18.5%	−1.1%
q_{100}	0.446	0.442	0.446	0.543	0.515	0.555	0.451
1950-60							
Life expectancy at age 75	7.20	2.3%	0.6%	−6.0%	1.3%	6.9%	6.0%
at age 85	3.85	−2.7%	−2.1%	−1.9%	−3.4%	−6.4%	−3.0%
at age 95	1.92	6.1%	6.4%	−5.9%	−4.1%	−12.9%	6.0%
q_{100}	0.572	0.455	0.459	0.562	0.526	0.564	0.459
Cohort, born 1875-1885							
Life expectancy at age 75	7.31	3.2%	1.6%	−5.2%	2.5%	8.5%	7.3%
at age 85	3.92	−1.9%	−0.8%	−0.6%	−2.2%	−5.2%	−1.9%
at age 95	2.06	−0.9%	1.0%	−10.6%	−9.5%	−18.4%	−0.4%
q_{100}	0.424	0.450	0.446	0.546	0.515	0.558	0.451
Cohort, born 1885-1895							
Life expectancy at age 75	7.44	3.2%	0.3%	−5.7%	2.1%	8.0%	6.9%
at age 85	4.00	−2.1%	−0.3%	−0.9%	−2.7%	−5.7%	−2.4%
at age 95	2.14	−1.8%	6.1%	−10.7%	−10.2%	−19.2%	−1.3%
q_{100}	0.424	0.446	0.412	0.536	0.509	0.553	0.446

* q_{100} from the crude data is estimated as average of rates q_{99}, q_{100} and q_{101} due to the very uneven development of the crude rates at these ages. Typically the rate at age 99 years was materially higher than age 100.

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

Table A.3. Evaluation of models, Irish females, aggregated over periods, by year of death and year of birth. Relative square error

	Kannisto	Logistic	Makeham	HP 1	HP 2	HP 3
1970-80						
Weighted least squares, ages 83-100	59	47	51	80	142	46
Weighted least squares, ages 75-100	135	144	613	169	429	146
(Unweighted) least squares, ages 88-98	83	101	37	44	288	121
1960-70						
Weighted least squares, ages 83-100	83	70	79	115	163	85
Weighted least squares, ages 75-100	190	171	508	229	528	487
(Unweighted) least squares, ages 88-98	137	188	82	91	314	145
1950-60						
Weighted least squares, ages 83-100	159	154	133	179	197	160
Weighted least squares, ages 75-100	336	318	337	389	843	625
(Unweighted) least squares, ages 88-98	312	274	190	200	389	349
Cohort, born 1875-1885						
Weighted least squares, ages 83-100	98	69	77	150	233	67
Weighted least squares, ages 75-100	399	213	506	495	1,268	309
(Unweighted) least squares, ages 88-98	104	134	41	62	397	127
Cohort, born 1885-1895						
Weighted least squares, ages 83-100	95	56	69	140	260	54
Weighted least squares, ages 75-100	263	161	707	308	957	190
(Unweighted) least squares, ages 88-98	88	55	10	39	481	54

Note: Values above have been rebased to aid comparability across the two distinct measures employed (weighted least squares and unweighted least squares).

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

Table A.4. Estimates of life expectancies, various ages, based on models, Irish females, aggregated over periods, by year of death and year of birth. Relative square error

	Based on data*	Kannisto	Logistic	Makeham	HP 1	HP 2	HP 3
1970-80							
Life expectancy at age 75	9.02	−0.9%	−2.6%	−10.7%	−1.3%	3.1%	−0.2%
at age 85	4.78	−3.0%	−0.7%	0.0%	−3.2%	−6.5%	−0.5%
at age 95	2.40	−1.0%	8.1%	−2.6%	−7.9%	−17.9%	10.3%
q_{100}	0.419	0.403	0.365	0.444	0.452	0.501	0.358
1960-70							
Life expectancy at age 75	8.63	0.2%	−2.0%	−9.5%	−0.3%	4.0%	4.0%
at age 85	4.60	−2.7%	−0.5%	−0.3%	−3.4%	−6.4%	−3.0%
at age 95	2.28	1.2%	11.7%	−2.5%	−7.1%	−16.1%	2.3%
q_{100}	0.439	0.411	0.368	0.464	0.467	0.510	0.407
1950-60							
Life expectancy at age 75	8.11	2.1%	1.9%	−5.8%	2.1%	6.4%	4.9%
at age 85	4.40	−3.2%	−3.6%	−2.3%	−4.3%	−7.2%	−3.0%
at age 95	1.98	13.6%	8.8%	2.1%	2.1%	−7.0%	17.2%
q_{100}	0.594	0.424	0.451	0.518	0.493	0.531	0.411
Cohort, born 1875-1885							
Life expectancy at age 75	8.35	3.9%	0.9%	−10.3%	3.7%	8.4%	2.6%
at age 85	4.64	−3.4%	−0.6%	0.3%	−4.4%	−7.6%	−0.5%
at age 95	2.42	−2.7%	12.5%	−1.7%	−11.6%	−20.7%	11.9%
q_{100}	0.403	0.412	0.348	0.444	0.473	0.519	0.354
Cohort, born 1885-1895							
Life expectancy at age 75	8.88	1.6%	−1.3%	−12.7%	1.0%	5.5%	0.0%
at age 85	4.86	−3.7%	−0.6%	0.5%	−4.3%	−8.5%	−0.3%
at age 95	2.59	−5.9%	10.0%	−2.7%	−13.0%	−24.1%	10.0%
q_{100}	0.377	0.401	0.333	0.420	0.451	0.511	0.337

*q_{100} from the crude data is estimated as average of rates q_{99}, q_{100} and q_{101} due to the very uneven development of the crude rates at these ages. Typically the rate at age 99 years was materially higher than age 100.

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

Table A.5. Evaluation of models, Irish males and females, aggregated over periods, by year of death and year of birth. Relative square error

	Kannisto	Logistic	Makeham	HP 1	HP 2	HP 3
Males						
1990-2000 (105%E)						
Weighted least squares, ages 83-100	6	5	8	31	86	6
Weighted least squares, ages 75-100	8	12	365	50	445	34
(Unweighted) least squares, ages 88-98	19	37	7	12	266	54
1990-2000 (100%E)						
Weighted least squares, ages 83-100	7	5	7	34	90	6
Weighted least squares, ages 75-100	8	10	346	61	463	41
(Unweighted) least squares, ages 88-98	19	38	7	13	281	51
1980-1990						
Weighted least squares, ages 83-100	17	18	18	39	92	29
Weighted least squares, ages 75-100	28	34	230	106	529	549
(Unweighted) least squares, ages 88-98	106	112	30	38	256	78
Females						
1990-2000 (105%E)						
Weighted least squares, ages 83-100	6	7	15	59	42	84
Weighted least squares, ages 75-100	9	9	18	269	205	2,128
(Unweighted) least squares, ages 88-98	3	11	74	53	141	181
1990-2000 (100%E)						
Weighted least squares, ages 83-100	6	7	17	72	47	89
Weighted least squares, ages 75-100	10	10	19	353	235	2,369
(Unweighted) least squares, ages 88-98	3	15	81	55	151	176
1980-1990						
Weighted least squares, ages 83-100	23	22	31	42	95	37
Weighted least squares, ages 75-100	79	69	421	68	218	56
(Unweighted) least squares, ages 88-98	33	34	28	22	265	67

Note: Values above have been rebased to aid comparability across the two distinct measures employed (weighted least squares and unweighted least squares).

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

Table A.6. Estimates of life expectancies, various ages, based on models, Irish males and females, aggregated over periods, by year of death. Relative square error

	Based on data*	Kannisto	Logistic	Makeham	HP 1	HP 2	HP 3
Males							
1990-2000 (105%E)							
Life expectancy at age 75	8.14	−0.6%	−1.1%	−9.3%	1.2%	5.9%	1.4%
at age 85	4.37	−0.8%	−0.1%	0.7%	−3.1%	−6.3%	0.1%
at age 95	2.32	2.6%	6.7%	−3.9%	−11.3%	−20.0%	8.9%
q_{100}	0.429	−0.6%	−1.1%	−9.3%	1.2%	5.9%	1.4%
1990-2000 (100%E)							
Life expectancy at age 75	8.09	−0.4%	−0.9%	−9.2%	1.5%	6.0%	1.6%
at age 85	4.36	−0.9%	−0.1%	0.5%	−3.3%	−6.8%	−0.1%
at age 95	2.36	0.5%	5.1%	−5.7%	−13.2%	−21.9%	6.7%
q_{100}	0.429	0.404	0.382	0.465	0.486	0.530	0.378
1980-1990							
Life expectancy at age 75	7.59	0.0%	0.6%	−7.6%	2.5%	7.4%	7.3%
at age 85	4.13	−0.8%	−1.0%	−0.3%	−3.1%	−5.9%	−3.0%
at age 95	2.13	7.7%	8.2%	−2.9%	−7.5%	−20.0%	1.2%
q_{100}	0.444	0.412	0.408	0.495	0.499	0.582	0.439
Females							
1990-2000 (105%E)							
Life expectancy at age 75	10.47	0.0%	−0.4%	−0.6%	2.7%	2.3%	6.5%
at age 85	5.45	−0.1%	−0.7%	−1.8%	−3.2%	−4.0%	−6.3%
at age 95	2.75	−0.8%	−5.7%	−13.9%	−17.5%	−16.5%	−16.3%
q_{100}	0.372	0.370	0.399	0.460	0.470	0.456	0.440
1990-2000 (100%E)							
Life expectancy at age 75	10.39	0.2%	−0.2%	−0.3%	3.5%	2.5%	7.3%
at age 85	5.43	−0.2%	−1.0%	−2.0%	−2.8%	−4.2%	−5.9%
at age 95	2.75	−1.0%	−6.5%	−14.4%	−17.4%	−17.0%	−16.3%
q_{100}	0.372	0.370	0.402	0.463	0.470	0.458	0.440
1980-1990							
Life expectancy at age 75	9.77	−2.4%	−2.1%	−7.6%	−1.2%	1.8%	−1.0%
at age 85	5.14	0.0%	−0.1%	−0.4%	−2.3%	−5.6%	−2.5%
at age 95	2.68	1.8%	1.7%	−9.3%	−10.4%	−21.6%	−11.1%
q_{100}	0.397	0.363	0.363	0.442	0.430	0.500	0.434

*q_{100} from the crude data is estimated as average of rates q_{99}, q_{100} and q_{101} due to the very uneven development of the crude rates at these ages. Typically the rate at age 99 years was materially higher than age 100.

Table A.7. Evaluation of models, Irish males and females, aggregated over periods, by year of death. Relative square error

	Kannisto	Logistic	Makeham	HP 1	HP 2	HP 3
Males						
2001-03 (105%E)						
Weighted least squares, ages 83-100	18	18	28	46	59	13
Weighted least squares, ages 75-100	36	35	225	115	297	136
(Unweighted) least squares, ages 88-98	34	59	128	131	344	31
2001-03 (100%E)						
Weighted least squares, ages 83-100	18	20	28	25	59	13
Weighted least squares, ages 75-100	40	30	200	40	303	145
(Unweighted) least squares, ages 88-98	32	60	137	74	323	32
Females						
2001-03 (105%E)						
Weighted least squares, ages 83-100	6	7	15	59	42	84
Weighted least squares, ages 75-100	9	9	18	269	205	2,128
(Unweighted) least squares, ages 88-98	3	11	74	53	141	181
2001-03 (100%E)						
Weighted least squares, ages 83-100	4	5	12	86	64	88
Weighted least squares, ages 75-100	11	29	38	286	250	1,972
(Unweighted) least squares, ages 88-98	11	14	63	64	117	177

Note: Values above have been rebased to aid comparability across the two distinct measures employed (weighted least squares and unweighted least squares).

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press

Table A.8. Estimates of life expectancies, various ages, based on models, Irish males and females, aggregated over periods, by year of death. Relative square error

	Based on data*	Kannisto	Logistic	Makeham	HP 1	HP 2	HP 3
Males							
2001-03 (105%E)							
Life expectancy at age 75	9.10	−0.9%	−1.0%	−5.1%	1.8%	3.9%	3.0%
at age 85	4.73	1.2%	0.2%	0.2%	−3.3%	−4.7%	0.3%
at age 95	2.74	−7.2%	−12.1%	−18.8%	−24.5%	−27.1%	−6.9%
q_{100}	0.378	0.385	0.413	0.476	0.495	0.504	0.381
2001-03 (100%E)							
Life expectancy at age 75	8.93	−1.2%	−0.1%	−4.9%	−0.7%	4.4%	3.2%
at age 85	4.65	1.2%	0.8%	0.1%	−0.6%	−4.1%	0.2%
at age 95	2.71	−7.0%	−12.3%	−19.4%	−16.7%	−26.6%	−7.4%
q_{100}	0.381	0.387	0.419	0.484	0.449	0.504	0.387
Females							
2001-03 (105%E)							
Life expectancy at age 75	11.35	0.0%	−0.4%	−0.6%	2.7%	2.3%	6.5%
at age 85	5.91	−0.1%	−0.7%	−1.8%	−3.2%	−4.0%	−6.3%
at age 95	2.96	−0.8%	−5.7%	−13.9%	−17.5%	−16.5%	−16.3%
q_{100}	0.331	0.370	0.399	0.460	0.470	0.456	0.440
2001-03 (100%E)							
Life expectancy at age 75	11.15	−0.6%	−1.3%	−1.6%	2.7%	2.9%	5.5%
at age 85	5.81	−0.2%	−0.2%	−1.3%	−2.1%	−2.1%	−5.6%
at age 95	2.92	−1.9%	−4.2%	−12.1%	−16.6%	−15.1%	−15.5%
q_{100}	0.333	0.356	0.372	0.427	0.445	0.430	0.416

https://doi.org/10.1017/S1748499500000609 Published online by Cambridge University Press